Law of the Jungle in a Linear Network

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Introduction

- Congestion control
  - Avoidance of packet loss
  - Adaptation of the throughput according to packet loss
Introduction

- Congestion control
  - Avoidance of packet loss
  - Adaptation of the throughput according to packet loss
- No Congestion control
  - No avoidance of packet loss
  - Sources send at their maximum rate: the access rate
  - Recovery mechanism from packet loss (source coding)

Is that sustainable? Is there any congestion collapse?
Law of the Jungle

Stability of a linear network

Theorem of anarchy
Flow Model

Congestion control model introduced by Massoulié and Roberts in 2000.

- Network modeled at a flow level, not a packet level
- Flows are going in the network like a fluid
- Users divided in classes to model heterogeneity of the traffic
- Dynamic traffic
- Resource allocation determined by congestion policy
Linear Network

- Two links of capacity 1
- 3 classes of flows
  - Class 0 going through both links with access rate 1
  - Class 1 going through link 1 with access rate 1
  - Class 2 going through link 2 with access rate $a$
- Flow generation rates: $\lambda_i$ (flows/sec)
- Flow size rates: $\mu_i$ (bits$^{-1}$)
- Traffic intensities: $\rho_i = \lambda_i/\mu_i$ (bits/sec)
Resource allocation

Congestion control

- Bandwidth allocation determined by Congestion control algorithm
- Input rate is equal to output rate
- Well studied examples: $\alpha$-fair allocations (proportional fairness, max-min fairness) (Mo and Walrand 2000)
Resource allocation

No congestion control

- Sources send at their maximum rate in the network
- Bandwidth allocation determined by buffer policy in the routers
- Output rate is different of input rate
- Fair policy: Fair Dropping (Shenker et al 98)
- Simplest policy: Tail Dropping (Law of the Jungle)
Tail dropping:
- Sources send at their maximum rate
- At each link, output rates are proportional to input rates
Resource allocation

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\[ \alpha = \frac{n_0}{n_0 + n_1} \]

\[ \phi_1 = \frac{n_1}{n_0 + n_1} \]
Resource allocation

Tail dropping:

- Sources send at their maximum rate
- At each link, output rates are proportional to input rates

\[
\alpha = \frac{n_0}{n_0 + n_1}
\]

\[
\phi_0 = \min(\alpha, \frac{\alpha}{n_2a + \alpha})
\]

\[
\phi_1 = \frac{n_1}{n_0 + n_1}
\]

\[
\phi_2 = \min(n_2a, \frac{n_2a}{\alpha + n_2a})
\]
We study the Markov process describing the number of flows in each class:

\[(N_0(t), N_1(t), N_2(t))\]

with transition rates:

\[n \mapsto n + e_i : \lambda_i\]
\[n \mapsto n - e_i : \mu_i \phi_i(n)\]
Performance metrics

- Mean flow response time
  - Ergodicity condition
- Optimal performance
  - The network can be stabilized if and only if
  \[ \forall l \sum_{i:l \in r_i} \rho_i < C_l \]
  - \( \alpha \)-fair policies are optimal (Bonald and Massoulié 2001, de Veciana et al 2001)
  - Fair dropping is optimal (Bonald et al 2009)
  - Tail Dropping is not optimal
Law of the Jungle

Stability of a linear network

Theorem of anarchy
Stability of class 2

We have $0 \leq \alpha \leq 1$. Thus, class 2 is stable if $\rho_2 < 1$ and transient if $\rho_2 > 1$ whatever the conditions on 0 and 1. Now, we suppose that $\rho_2 < 1$
Stability of class 2

We freeze the number of flows in classes 0 and 1 and then $\alpha$. There is a stationary distribution $\pi^\alpha$ for class 2. We then define the averaged throughput of class 0:

$$\bar{\phi}_0(\alpha) = \sum_{n_2 \in \mathbb{N}} \pi^\alpha(n_2) \min \left( \alpha, \frac{\alpha}{n_2 a + \alpha} \right)$$
Fluid limit of the system

Class 2 is stable, so there is no need to scale it! We perform the scaling only on classes 0 and 1:

\[(N_0(0), N_1(0)) = n_k, \quad \lim_{k \to \infty} \|n_k\| = \infty,\]

\[\lim_{k \to \infty} \frac{1}{\|n_k\|} (N_0(\|n_k\| t), N_1(\|n_k\| t)) \overset{d}{=} (z_0(t), z_1(t))\]

and \((z_0(t), z_1(t))\) satisfies:

\[
\dot{z}_0(t) = \lambda_0 - \mu_0 \phi_0 \left( \frac{z_0(t)}{z_0(t) + z_1(t)} \right)
\]

\[
\dot{z}_1(t) = \lambda_1 - \mu_1 \frac{z_1(t)}{z_0(t) + z_1(t)}
\]

The proof is similar to the one of Hunt and Kurtz in 1994 about Loss Networks.
Fluid Limit of the system

\[ \dot{z}_0(t) = \lambda_0 - \mu_0 \bar{\phi}_0 \left( \frac{z_0(t)}{z_0(t) + z_1(t)} \right) \]

\[ \dot{z}_1(t) = \lambda_1 - \mu_1 \frac{z_1(t)}{z_0(t) + z_1(t)} \]

There is a separation of time scale between the fluid limit \((z_0, z_1)\) and class 2: class 2 is always at equilibrium and there is an averaging on class 2 for fluid limit \(z_0\).
Fluid Limit dynamics

\[ \frac{z_0}{z_0 + z_1} = \phi_0^{-1}(\rho_0) \]

\[ \dot{z}_0 = \lambda_0 - \mu_0 \bar{\phi}_0 \left( \frac{z_0}{z_0 + z_1} \right) \]
Fluid Limit dynamics

\[ \dot{z}_1 = \lambda_1 - \mu_1 \frac{z_1}{z_0 + z_1} \]
Fluid Limit dynamics

\[
\frac{z_0}{z_0 + z_1} = \phi_0^{-1}(\rho_0)
\]

\[
\frac{z_1}{z_0 + z_1} = \rho_1
\]

\[
\rho_0 < \phi_0(1 - \rho_1)
\]
Fluid Limit dynamics

\begin{align*}
\frac{z_1}{z_0 + z_1} &= \rho_1 \\
\frac{z_0}{z_0 + z_1} &= \phi_0^{-1}(\rho_0)
\end{align*}

\( \rho_0 > \phi_0(1 - \rho_1) \)
Stability Conditions

The exact conditions for stability under the Law of the Jungle are:

\[ \rho_1 < 1, \quad \rho_2 < 1, \]
\[ \rho_0 < \bar{\phi}_0(1 - \rho_1) \]

The optimal stability conditions are:

\[ \rho_1 < 1, \quad \rho_2 < 1, \]
\[ \rho_0 < \min(1 - \rho_1, 1 - \rho_2) \]

But:

\[ \bar{\phi}_0(1 - \rho_1) < \min(1 - \rho_2, 1 - \rho_1) \]

The stability conditions are not optimal!
Law of the Jungle

Stability of a linear network

Theorem of anarchy
What is the anarchy?

The price of anarchy quantifies how far the stability conditions are from optimal ones. In our case, the definition is simply:

\[ P(a) = \max_{\rho_1, \rho_2} \left( \min(1 - \rho_1, 1 - \rho_2) - \bar{\phi}_0(1 - \rho_1) \right) \]

The price of anarchy depends on \( a \), the access rate of class 2.
Theorem of anarchy

\[ \phi_0(1 - \rho_1) \]

For any \((\rho_0, \rho_1, \rho_2)\) satisfying the optimal stability conditions, there exists a small enough such that the network is stable.

\[ \lim_{a \to 0} P(a) = 0 \]
Conclusion

- Do we need congestion control in the Internet?
- Evaluation of the impact of big clients on the network (optical networks).

- A fluid limit with an interesting averaging phenomenon
- Can be extended to linear networks with more than two links
- In other contexts?

- **Conjecture**: The theorem of anarchy is true in acyclic networks.