The time scales of a stochastic network with failures

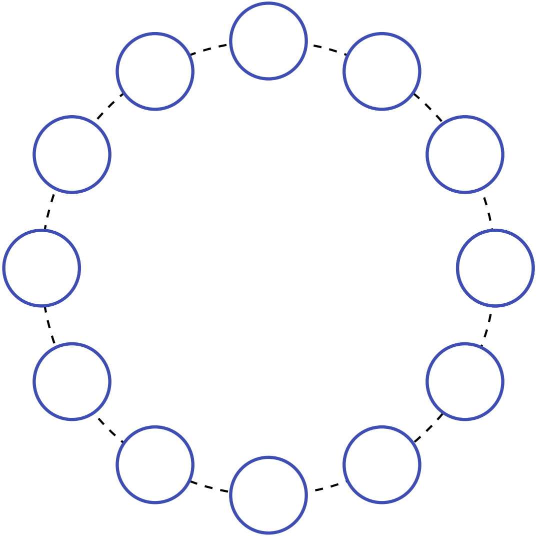
Mathieu Feuillet

joint work with Philippe Robert

YEQT-V

Contents

An inspiring example: a DHT



For a specific file

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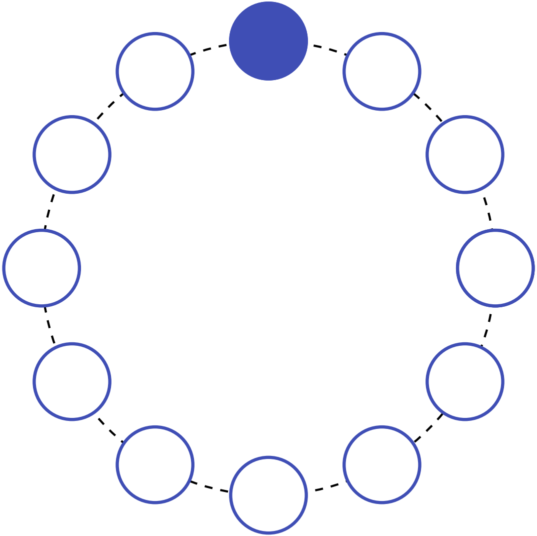

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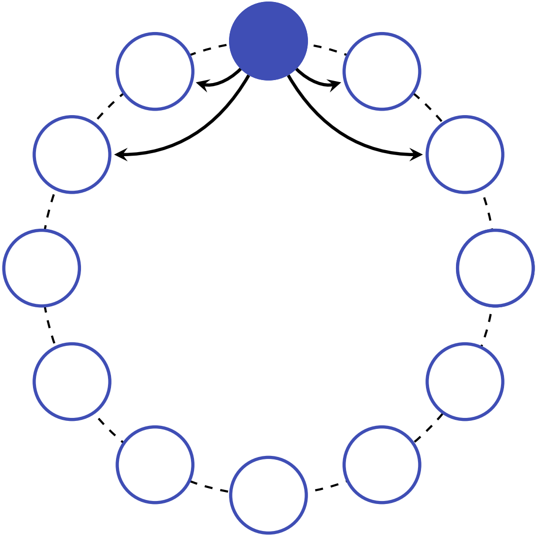

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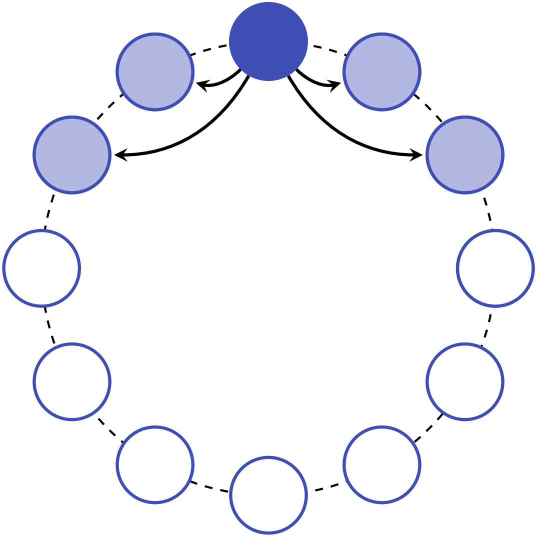

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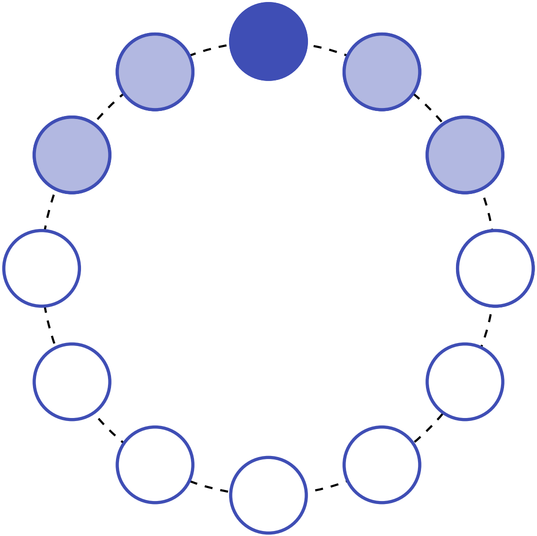

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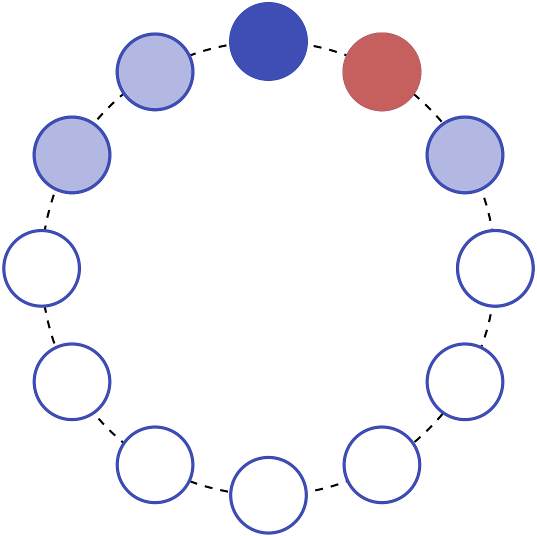

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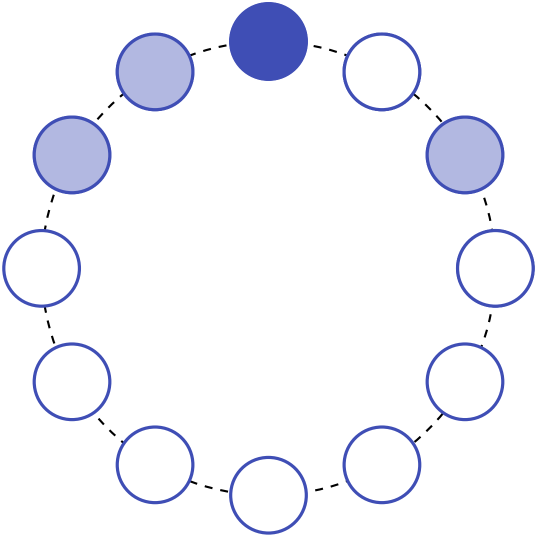

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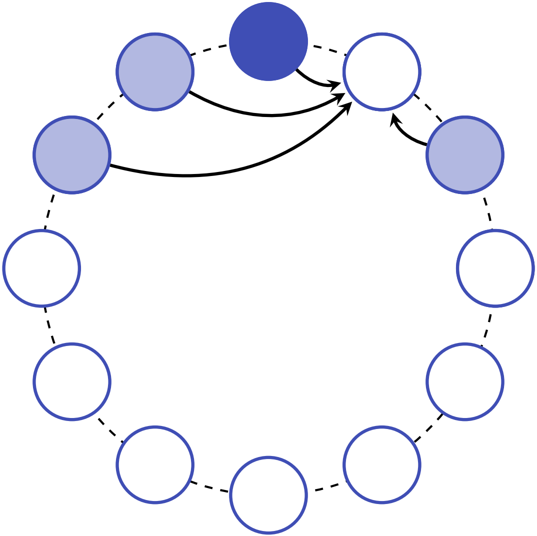

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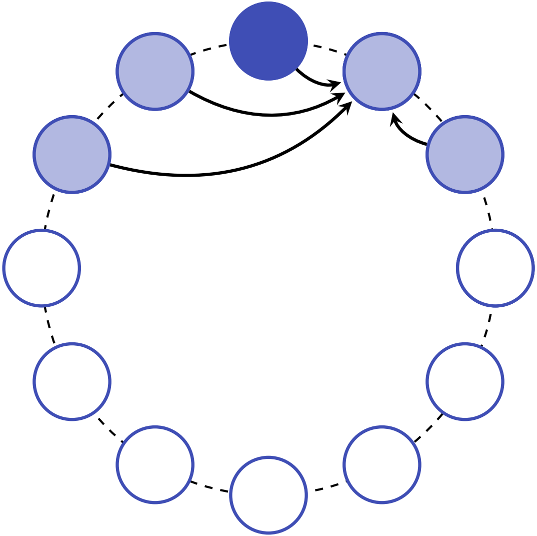

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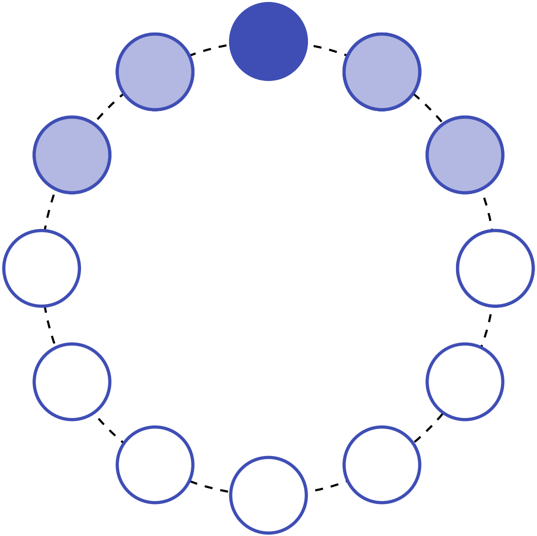

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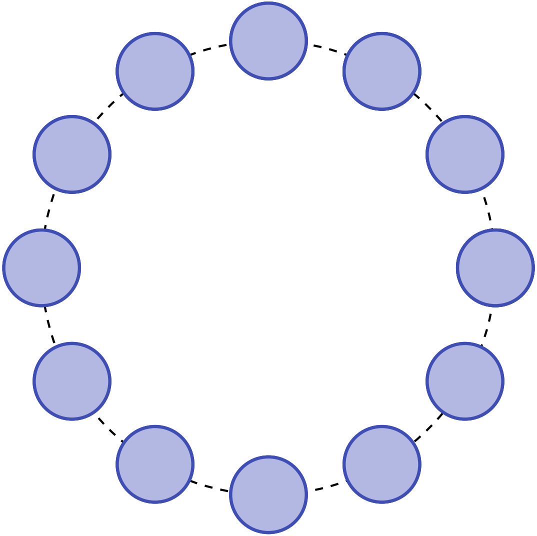

From a global perspective

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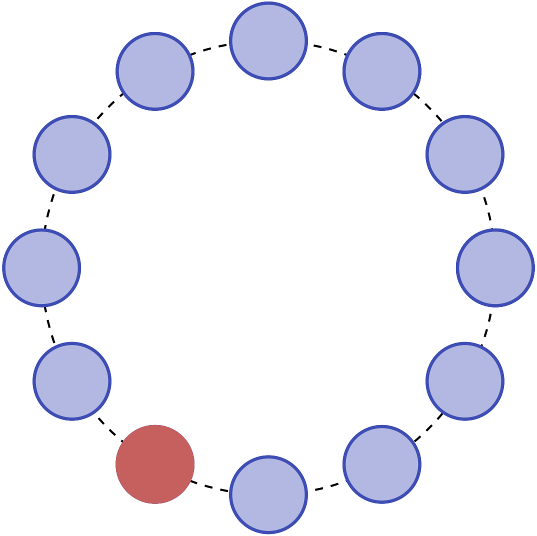

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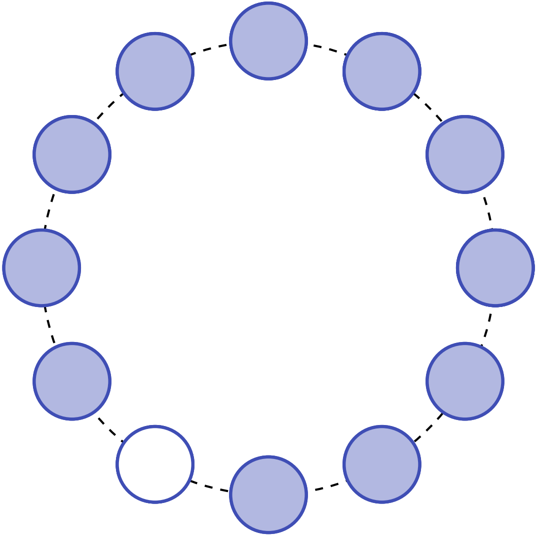

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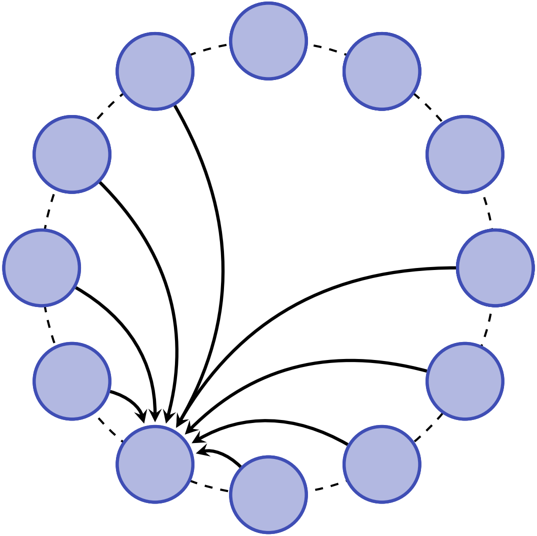

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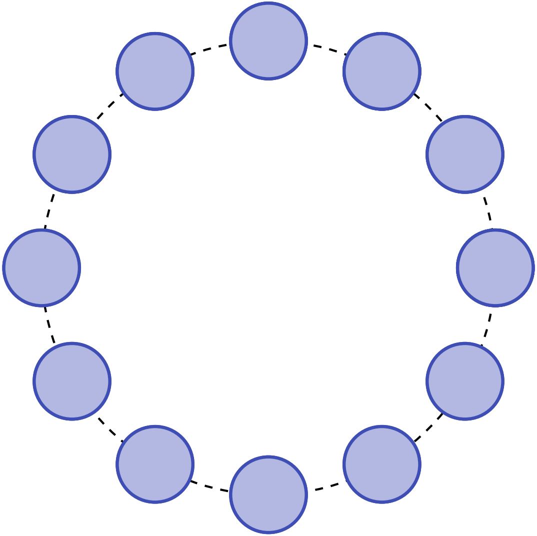


From a global perspective

An inspiring example: a DHT

Typical questions:

What is the maximum number of files that the system can sustain?

What is the decay rate of the network?

General problem:
Study the evolution of a large distributed system with failures.

Background

- Reliability theory:
  System Reliability Theory: Models, Statistical methods and Applications, Raudand, Hoyland,2003.

  Very few queueing analysis of large systems.

- Statistical studies:

  fta.inria.fr

Contents

Model

- A set of F_N files.
- At most d copies per file.
- Each copy is lost at rate μ.
- Capacity of duplication: λN.
- A file with 0 copies is lost.

Model

For $0 \leq i \leq d$:
$X_i(t)$: number of files with i copies.

Back-up policy:
Recovery capacity λN allocated to the files with the minimum number of copies.

$(x_i, x_{i+1}) \rightarrow (x_i - 1, x_{i+1} + 1)$ at rate λN if
$x_1 = x_2 = \cdots = x_{i-1} = 0$ and $x_1 > 0$.

Model

$(X_i(t), 0 \leq i \leq d)$ is a transient Markov Process.

$$X_0(t) + X_1(t) + \cdots + X_d(t) = F_N.$$

Unique absorbing state $(F_N, 0 \ldots, 0)$.

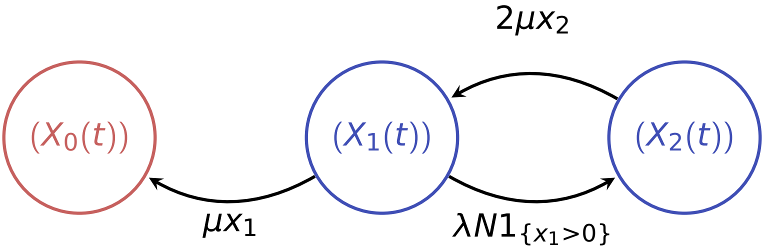
Assume

$$\lim_{N \to \infty} \frac{F_N}{N} = \beta > 0.$$

General problem:

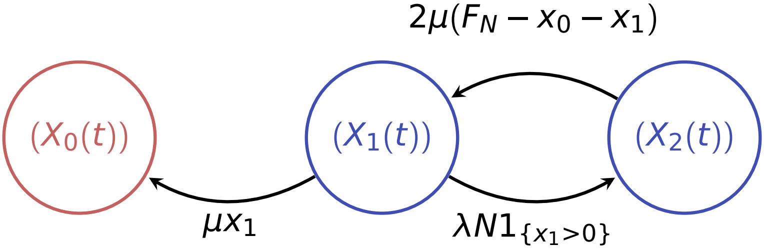
Estimate the decay rate of the network.

Case $d = 2$



Case $d = 2$

State $(x_0, x_1, F_N - x_0 - x_1)$.



Different behaviors

Three time scales:

$$\begin{cases} t & \to t/N \\ t & \to t \\ t & \to Nt \end{cases}$$

Three regimes:

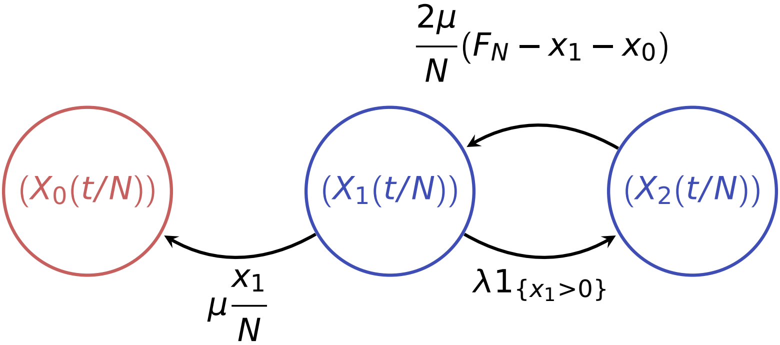
Overloaded network: $2\beta > \rho = \lambda/\mu$.

Critical case: $2\beta = \rho$.

Underloaded case: $2\beta < \rho$.

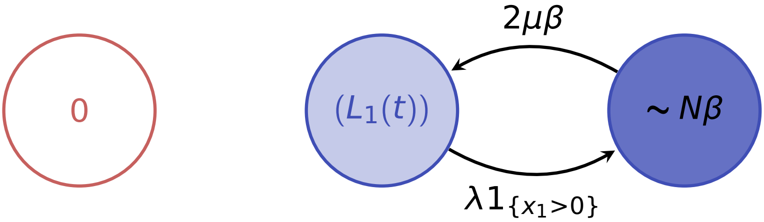
Contents

Time scale: $t \to t/N$



Time scale: $t \to t/N$

$$(L_1(t))\text{: an } M/M/1 \text{ queue } \begin{cases} \text{ergodic if } 2\beta < \rho, \\ \text{null recurrent if } 2\beta = \rho, \\ \text{transient if } 2\beta > \rho. \end{cases}$$



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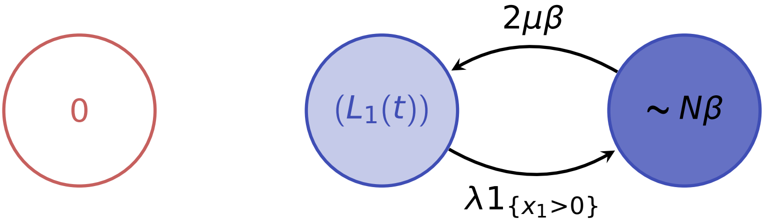


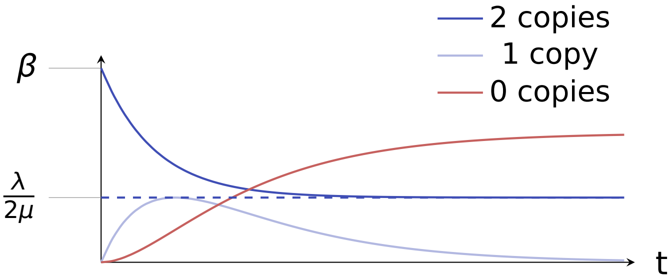
No loss!

Contents

Overloaded network

If $2\beta > \rho$, $(X_0(t)/N, X_1(t)/N, X_2(t)/N)$ converges to

$$\begin{cases} x_0(t) &= (\beta - \rho/2)(1 + e^{-2\mu t} - 2e^{-\mu t}), \\ x_1(t) &= (2\beta - \rho)(e^{-\mu t} - e^{-2\mu t}), \\ x_2(t) &= (\beta - \rho/2)e^{-2\mu t} + \rho/2. \end{cases}$$



Technical point: Generalized Skorokhod problem.

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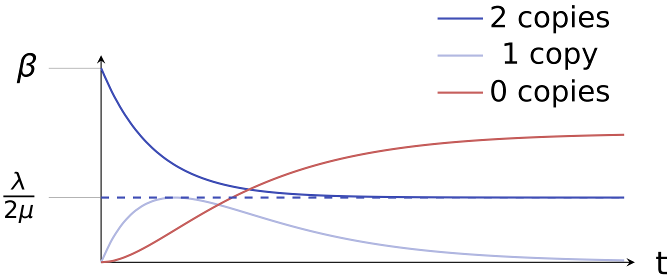
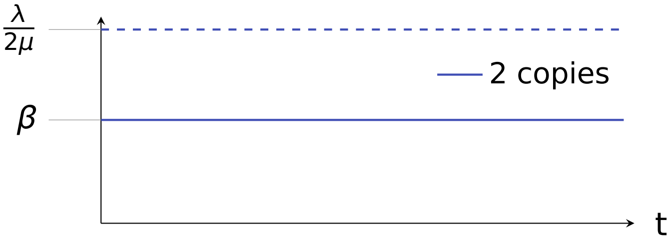


Some losses!

Underloaded network

If $2\beta \leq \rho$, $(X_0(t)/N, X_1(t)/N, X_2(t)/N)$ converges to

$$\begin{cases} x_0(t) & = 0, \\ x_1(t) & = 0, \\ x_2(t) & = \beta. \end{cases}$$



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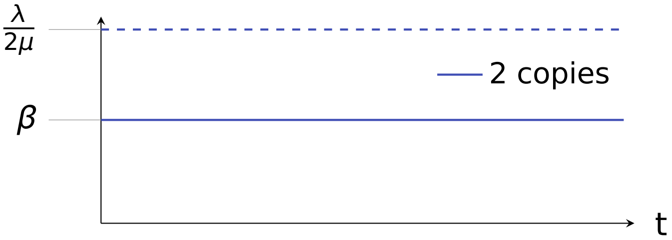


No loss!

The critical case

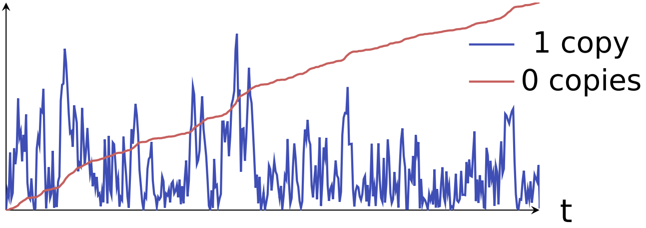
If $2\beta = \rho$,

$$\left(\frac{X_0^N(t)}{\sqrt{N}}, \frac{X_1^N(t)}{\sqrt{N}}\right) \Rightarrow \left(\int_0^t Y(u)\,\mathrm{d}u, Y(t)\right)$$

where

$$\mathrm{d}Y(t) = \sqrt{2\lambda}\,\mathrm{d}B(t) + \mu\left(2\gamma - 3Y(t) - 2\mu\int_0^t Y(u)\,\mathrm{d}u\right)\mathrm{d}t$$

with the constraint $Y(t) \geq 0$.

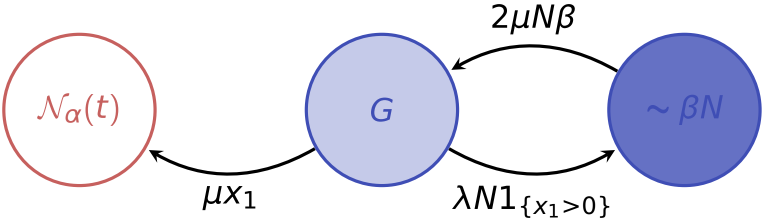


Underloaded network

If $2\beta < \rho$, $X_2(t)/N \Rightarrow \beta$

$X_1(t) \Rightarrow G$ Geometric r.v. w. param. $2\beta/\rho$,

$(X_0(t)) \Rightarrow (\mathcal{N}_\alpha(t))$ with $\alpha = 2\mu\beta(\rho - 2\beta)$.



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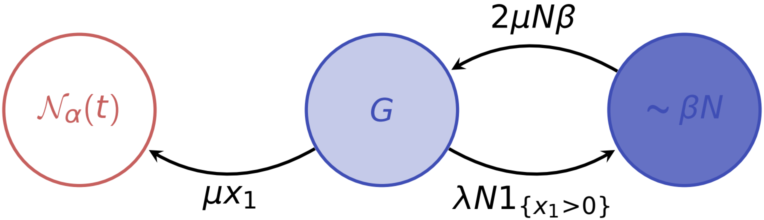


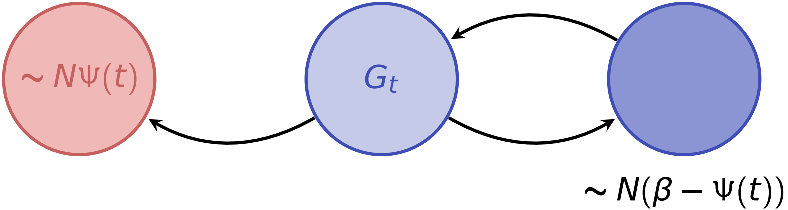
No significant losses!

Contents

Time scale $t \to Nt$

$$\lim_{N\to+\infty}\left(\frac{X_0(Nt)}{N}\right)=\Psi(t),$$

G_t Geometric r.v. with par. $2(\beta-\Psi(t))/\rho$



Stochastic averaging: At “time” Nt, X_1 behaves as an $M/M/1$ process at equilibrium:

$+1$ at rate $2\mu(\beta-\Psi(t))$

-1 at rate λ.

Time scale $t \to Nt$

$$\lim_{N\to+\infty}\left(\frac{X_0(Nt)}{N}\right)=\Psi(t),$$

where $\Psi(t)$ unique solution of

$$\Psi(t)=\mu\int_0^t \frac{2\mu(\beta-\Psi(s))}{\lambda-2\mu(\beta-\Psi(s))}\,\mathrm{d}s.$$

Time scale $t \to Nt$

$$\lim_{N\to+\infty}\left(\frac{X_0(Nt)}{N}\right)=\Psi(t),$$

where $\Psi(t)$ unique solution in $(0,\beta)$ of

$$(1-\Psi(t)/\beta)^{\rho/2}\,e^{\Psi(t)+t}=1.$$

As $t\to\infty$ then $\Psi(t)\sim\beta-e^{-2(\beta+t)/\rho}$.

$t\to Nt$ "correct" time scale to describe decay.

Decay rate of the network

$$T_N(\delta) = \inf\{t \geq 0 : \; X_0^N(t) \geq \delta N\}$$

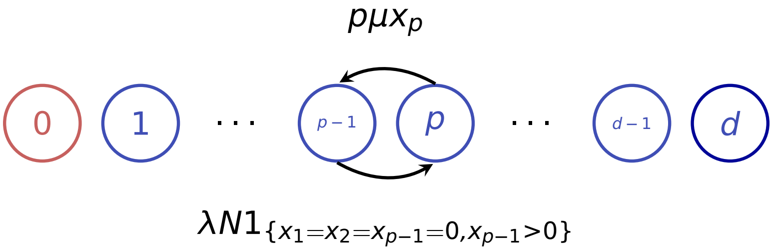
Theorem:

$$\lim_{N\to\infty} \frac{T_N(\delta)}{N} = -\frac{\rho}{2}\log(1-\delta) - \delta\beta.$$

Contents

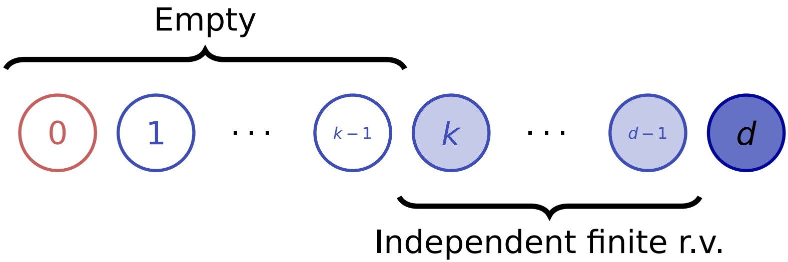
Case $d \geq 3$

If $\rho > d\beta$
d time scales: $t \to N^k t$, $0 \leq k \leq d-1$,



Time scale of decay: $t \to N^{d-1} t$

At time scale $t \to N^{d-k+1}$.



A cascade of time scales.

Conclusion

- A very simple but rich model.
- A rule of the thumb for dimensioning:
  Trade-off between
  * the capacity $d\beta < \rho$
  * the decay rate of order N^{d-1}.

Future research:

- Distributed back-up mechanism
- Geometrical considerations

Thank you!

Annexes

Technical corner: Skorokhod

Main difficulty: discontinuity of $\lambda N \mathbb{1} x_1 > 0$.

Technical corner: Skorokhod

Main difficulty: discontinuity of $\lambda N \mathbb{1} x_1 > 0$.

Usual solution: Skorokhod problem.

$$X(t) = Z(t) + R(t)$$

Constrained process

Pushing process

Free process

Conv. of $(Z(t))$ + Skorokhod

$$\Downarrow$$

Convergence of $(X(t))$

Technical corner: Skorokhod

Main difficulty: discontinuity of $\lambda N \mathbb{1}x_1 > 0$.

Usual solution: Skorokhod problem.

$$\underbrace{X(t)}_{\text{Constrained process}} = \underbrace{Z(t)}_{\text{Free process}} + \underbrace{R(t)}_{\text{Pushing process}}$$

Does not apply here!

Technical corner: Skorokhod

Main difficulty: discontinuity of $\lambda N \mathbb{1} x_1 > 0$.

Our approach: Generalized Skorokhod problem.

$$X(t) = G(X)(t) + R(t)$$

Constrained process

Functional

Pushing process

Conv. of free equation + Skorokhod

$\Downarrow$

Convergence of $(X(t))$